Can using concrete manipulatives help to develop relational/conceptual understanding?

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I hear and I forget.
I see and I remember.
I do and I understand.
-Confucius (551-479 BC)

Introduction and Literature Review

I have noticed over the years that students often see maths as a set of steps and rules that they apply in certain situations without fully understanding what they are doing or why. One of the things that interests me most as a teacher is how I can ensure that the students I teach are developing a relational or conceptual understanding. Even after what I feel is a good explanation of a topic there continue to be nonsensical errors from a number of students. Skemp (1976) discusses the difference between relational and instrumental understanding and the benefits of developing a relational understanding. He defined the two different types of understanding: instrumental understanding as “rules without reasons”; and, relational understanding as ‘knowing what to do and why”. He thought that the ability to perform algorithms themselves was useful, but ensuring that they are not taught in a didactic manner, which could remove the value and relevance of the procedure, was the key. This relational understanding then enables students to make links between topics and enables them to apply their knowledge to unfamiliar situations. What I understand by instrumental understanding is learning by rote. The Oxford English Dictionary definition of ‘by rote’ is “in a mechanical manner, by routine; especially by the mere exercise of memory without proper understanding of, or reflection upon, the matter in question”. This to me sums up how algorithms and methods are frequently used by my students. One example that comes to mind is: ‘take it over to the other side and change the sign’ when solving equations.

Dienes (1916 -2014), Paiget (1896 – 1980) and Montessori (1870 – 1952) have all advocated the use of concrete materials in the formation of understanding. As a result I started to look at how I could use manipulatives within the classroom to help develop relational understanding. Physical objects have been traditionally used in primary classrooms to introduce abstract concepts such as quantity and fractions.

Algebra is an abstract concept that many students struggle with so I felt that it would be an area ideal for the application of manipulatives.

Current research refers to procedural knowledge and conceptual understanding and from the work of Star (2005 and 2007) and Baroody, Feil and Johnson (2007) on interpretations of these terms it is clear that they are not mutually exclusive. The relationship between the two and how they reinforce each other was interesting and I could relate it to my own mathematical learning. When I am first introduced to a new topic I feel that I develop both procedural (instrumental) and basic conceptual understanding but as I delve more deeply into the topic and question myself, my relational understanding develops and I gain a deeper conceptual understanding. I see this as promoting the fact that some procedural knowledge is needed in mathematics and a proficiency in procedures can help to develop a conceptual or relational understanding. Or as Reason stated *"I feel the need to help pupils acquire both instrumental understanding and relational understanding of mathematics, rather than one or the other…Skills are often needed before things can be understood relationally* ". (Reason, 2003 pp 6). Similarly Skemp felt that "*even relational mathematicians often use instrumental thinking*" (Skemp, 1976 pp25), which backs up my feelings about how I learn and use mathematics.

Skemp (1983) also felt that Mathematics is still often taught as a pen and paper exercise where students work alone and are not encouraged to question what they have been told. Rather than teach maths as a set of instructions and rules, I feel that it should be taught in a manner that allows the students to question their learning and provides them with a number of experiences to challenge their understanding. Rosnick and Clement (1980) believe that it is essential that students understand the concept of a variable, but as it is a simple but abstract concept it is difficult to teach.

Dienes (1960) discusses 4 stages in learning mathematics; Dynamic Principle, Constructivity Principle, Mathematical Variability Principle and the Perceptual Variabiliy Principle. The Dynamic and Constructivity Principles look at the use of structured games and concrete materials, which will precede analysis, concept-formation and abstraction. When looking at algebra it is important that the students understand the principle of a variable, and this is done by making students conscious of what happens to different numbers in the same situation (Mathematical Variability Principle).

Finally the Perceptual Variability Principle *" demands a richness of concrete experiences with the same conceptual structure, so that again all children can glean the essentially abstract idea inherent in any particular formula*." (Dienes, 1960, pp. 64)

However, Skemp noted that, *"in order to give someone a particular concept, you must do two things: arrange for him a set of experiences which have the concept in common; and, if it is a secondary concept, it is also essential to make sure that he has the concepts from which it is derived*". (Skemp. 1962, pp 14)

Algebra tiles are concrete manipulatives, which can be used to develop conceptual understanding. These can be used as a bridge to help develop abstract reasoning. They allow students to use concrete objects to observe, model and internalise abstract concepts (see appendix 3).

In terms of developing construction and higher abstraction, Leitze and Kitt (2000) discuss the use of algebra tiles as concrete models to introduce concepts. They note that they had seen successes in the learning of Algebra with these tiles. They believe that they "*reach a broader group of students by sequencing instruction from the concrete level, through the pictorial level, and finally to the abstract – or symbolic – level. The algebra tiles give a frame of reference to students who are not abstract thinkers*." (pp 463).

I asked my class the following two questions,

1. I think of a number, then add 5 and my answer is 8. What was my number?

2. Solve the equation $x + 5 = 8$.

Although these are essentially the same question, every single student in the group of six got the correct answer to question 1, with only two students getting the correct answer for question 2. This demonstrates that the students are able to think logically and that the use of symbols and formalism makes mathematics seem like something new rather than something that the student already knows. (Bruner 1960). Mathematics is a logical discipline and it has certain rules. We often proceed to algorithms that make it appear that what we are doing has no bearing on how one would proceed by non-rigorous routes.

Leitze and Kitt go on to descried the specific use of the tiles *“When most algebraic concepts are introduced, neither the teacher nor students should do abstract work but should rely on the tiles to solve problems and answer questions. Students should just manipulate the tiles and write the answers to the problems. At the next stage, students should manipulate the tiles and draw a brief sketch. The sketch should not be tedious. Eventually, the sketches will become mental visual representations that enable students to understand paper-and-pencil manipulations of algebraic symbols.”* (Leitze and Kitt, 2000 pp 463)

I decided to follow this approach and ensure that the initial work was relying on the tiles and that there was no abstract work, or any need to write down a method. The emphasis of the research being that the students should be working on Mathematics, rather than simply remembering what to do.

Context

I teach mathematics at a large independent boarding school in the East Midlands. There are approximately 1100 students in the age range 10-18. Class sizes are relatively small and I conducted this research with a class of six year 7 lower achieving students. I chose to work with this group because they are not confident with algebra and also the small number of students made it easier and quicker to conduct the stimulated recall interviews.

Research Methods

This research took place over a period of 3 months through multiple choice tests and stimulated recall interviews. There is a current focus in school on learning, and the students have all been involved in a number of lectures and talks on how they learn. As a result of this I decided to work collaboratively with the students, involving them in any decisions and discussing the use of the algebra tiles.
I was focussing on Classroom Action Research with some Participatory Action Research (Kemmis and McTaggart, 2005) getting views of the students throughout the research project. I also used the participant observation method to collect data by observing the students, listening and enquiring during the process.

The class had all already encountered basic algebra and the idea of a variable, and had some knowledge of simplifying and solving equations. I decided to obtain some quantitative data by giving the group a test (see appendix 1) on this before I started the intervention with the tiles. Following the test I conducted some stimulated recall interviews (qualitative data) with the students to try to get an insight into their thought processes when completing the questions. Prompts were used judiciously to seek further information rather than to lead. Each interview was tape recorded and fully transcribed.

I then looked at simplifying and solving equations with the group using the algebra tiles. A few were slightly reluctant at first, thinking that they were slightly babyish, but after a few lessons the whole class were on board with them and were able to confidently manipulate the tiles and simplify expressions. Following the teaching of this in lesson time, I then gave the group a further test (see Appendix 2) and conducted further stimulated recall interviews. This second test was delayed by several weeks following the teaching using the algebra tiles; this was to try to avoid any skewed results, as the first test had been taken several weeks after their first introduction to the topics.

The two tests were similar, but not identical, having the questions in a different order and with slightly different numbers. This was to ensure that the students had not simply memorised the answers whist ensuring that the questions were of the same difficulty. The multiple-choice answers were specifically chosen to identify common misconceptions that the students may have.

Ethics

During this research the class were aware that they were being observed and all parents had received a letter from the school to inform them of this.
The students were aware that they could stop participating at any time, and did not have to take part in the stimulated recall interviews.
All names and other information about the students have been kept strictly confidential.

Results and Insights

Table 1 summarises the test results for each student pre- and post intervention.

Table 1: The percentage results for each student in each of the two tests.

Student	1	2	3	4	5	6
Test 1	48	48	48	28	32	52
Test 2	59	59	47	29	47	76

Whilst all students showed some improvement in test score (albeit minimally in the case of student 4), my main question was on the conceptual understanding of the students, and this is where the stimulated recall interviews were useful.

Simplifying Expressions

Too begin with I will look at the questions on simplifying and compare the before and after interviews.

Before Intervention Interviews

None of the students answered question one correctly,

1. Simplify $2x - 4m + 3x - 2m$

A $5x + 6m$ **B** $11xm$ **C** $-x - 6m$ **D** $5x - 6m$

The most popular answer was B, and when interviewing student 5 their reasoning was;

Student 5 - I simplified it.

Me – How?

Student 5 - I saw how many like x's there were and added them together and then simplified it.

Two errors appear to have been made: 1) the negative signs have been ignored and the student has just added up all the numbers to 11 and 2) they have put the two variables together making *xm*. The latter is what Tall and Thomas (1991) call the “expected answer”; students are not happy to leave an operator in the answer so continue to simplify.

Only one student in the group got the answer to question 2 correct,

2. Simplify $a+b+a-b$

A $2ab$ **B** $2a+2b$ **C** $2a$ **D** $a-b$

But when interviewed the response was;

Student 5 - Because 2a minus b plus b, I don’t really know that one I just kind of guessed it.

So although the answer was correct, no understanding was shown.

Another popular answer was A, where a typical student response was;

Student 2 - I added them together and then I took one b away.

This time the negatives were taken into account, but the numbers were just added to get 2 and the variables put together to give *ab*.

Another simplifying question, number 6, again resulted in only one correct answer,

6. Simplify $3x+4+5x-6$

A $8x+10$ **B** $8x-2$ **C** $6x$ **D** $-2x-2$

One of the incorrect answers was explained as below;

Student 3 – So I got 3 and then I think I added it to 4 which makes 7 and then I plussed 5 which is 13, yeah 13 and then I minussed 6, but I don’t know how I got that answer. yeah, because I got 3 and 5, minus 3 from 5 and you get, no minus 5 from 3 which is minus 2x and then minus 6 from 4 which is minus 2.

From considering all of the responses in the before interviews it was clear that there was a very limited, superficial understanding of simplifying expressions from the students. They appeared to have a procedural approach to the questions but were then unable to apply their procedure accurately.

After Intervention Interviews

The following is an example of a simplifying question on the second assessment test.

6. Simplify $2x + 3 - x - 5 + 3x$

A $6x + 8$ B $4x - 2$ C $6x + 2$ D $4x + 8$

Most students answered this question successfully and the explanations generally demonstrated a better understanding, although there were still a few students who seemed unsure and said they had guessed.

One example of a response using the tiles is;

Student 2 - I took two of the x tiles and three of the little ones and then I got a negative x and three little red tiles and another 3 x tiles, so I did the x's first so I found the answer for the x and then I did the numbers.

This student was the most proficient at using the tiles and demonstrated the answer during the interview with the tiles. The tiles seem to have helped her separate the different variables, and the variables from the numbers. Thus avoiding the "expected answer" mistake from the previous test.

Student 5 - you add up how many x's and how many y's, I think I guessed that question, I don't get it with the minuses.

So although student 5 scored well on the test (having the highest added value in the group) and got the correct answer, the explanation of what was being done was still lacking in understanding. With her even adding in a y variable where none existed. This seems to imply a lack of understanding of a variable, and as other questions involved two variables (x and y) she applied the same logic and convention to this question.

Solving Equations

The other type of question that the students were given was solving simple linear equations.

Before Intervention Interviews

The following question was answered correctly by 4 of the 6 students in the first test;

16. Solve the equation $x + 6 = 10$

A $x = 3$ **B** $x = 16$ **C** $x = 4$ **D** $x = -4$

One of the students who answered incorrectly gave this explanation, for their answer B;

Student 4 - because 6 equals, I'm sorry, like how do I put this. Because 6 plus 10 equals 16 so you can just do the opposite and it will give you x equals 16.

This to me sums up the procedural process of moving to the other side and changing the sign. Unfortunately without conceptual understanding, mistakes like above are likely to occur, where the 6 has been added to the 10, rather than subtracted.

Student 5 - because I took 6 away from 10 so I thought that x would be 4 because 4 plus 6 would be 10.

This student clearly understands the concept of a variable and is using a logical thought process to work out the correct answer.

After Intervention Interviews

Five students in the second test answered the following question correctly;

16. Solve the equation $x + 3 = 5$

A x = 8 **B** x = 5 **C** x = 3 **D** x = 2

Some of the students who answered correctly gave the following explanations;

Student 2 - I got one of the x tiles and three of the little tiles and 5 of the other ones and then I added, no wait I took three from there so that means three and three so I was left with 2.

While she was talking, she was demonstrating with the tiles. Putting one x tile and three unit tiles on one-side ad 5 unit tiles on the other. She then was able to balance the equation by doing the same thing to both sides (taking away three tiles) and showing that x was equal to two.

It was clear from the demonstration and explanation that student 2 was confident at balancing equations and was doing the same thing to both sides. This shows a conceptual understanding of the idea of an equation, and she was competent at manipulating the tiles to show her working out.

Student 3 - I think I used the tiles, no I didn't I used 5 and then I minussed 3 and it was 2

Student 3 was becoming more confident at working out the answers without using the tiles, although there is a hint of a procedural approach of changing the side and changing the sign.

Quantitative Data

Although the sample size is very small it is possible to conduct a t-test with the data (de Winter, 2013) - summarised in Table 2 below.

Table 2: T-test data for the student test data presented in Table 1

Student	1	2	3	4	5	6
Test 1	48	48	48	28	32	52
Test 2	59	59	47	29	47	76
d	11	11	-1	1	15	24

$$H_0 : \mu_d = 0 \qquad H_1 : \mu_d > 0$$

Significance level 0.05

$$\upsilon = 6 - 1 = 5$$

Critical value $t_5(0.05) = 2.015$

$$\frac{\sum d}{n} = \frac{61}{6} = 10.167$$

$$s^2 = \frac{\sum d^2 - n\bar{d}^2}{n-1} = \frac{1045 - 6(10.167)^2}{5} = 84.9586$$

$$t = \frac{\bar{d} - \mu_d}{\frac{s}{\sqrt{n}}} = \frac{10.167 - 0}{\sqrt{\frac{84.9586}{6}}} = 2.702$$

This is significant therefore we can reject the null hypothesis H_0. Therefore the mean of the second test is greater than the mean of the first test, and there is less than a 5% chance that this could have occurred by chance.

This has shown that using the Algebra Tiles to teach simplifying and solving linear equations has improved the student's score on the test.

These results should be used cautiously. Obviously this was a very small sample of only six students, and also there is no guarantee that the improvement is directly related to the tile use. Had I just taught the topic again in an abstract manner, the repetition could have also caused increased marks in the assessment.

The aim of using the tiles was to provide a concrete illustration for something abstract. All students began using the tiles and as we progressed through the topics, they decided if they were ready to progress to the quick sketch and finally the pencil and paper methods. The students were all at different stages in their learning and moved at different paces through the stages. This also makes analysis difficult, and during the interviews the students who were no longer using the tiles appeared to have an instrumental way of describing what they were doing. It was therefore difficult to assess their level of conceptual understanding. Some were quick to say they had guessed or they didn't know how they got their answer as I feel they may have felt embarrassed or worried that they would get it wrong.

Table 3 below shows if the students used the tiles when completing the second test.

Table 3: Pre and post test data with summary of whether tiles used in test two

Student	1	2	3	4	5	6
Test 1	48	48	48	28	32	52
Test 2	59	59	47	29	47	76
Tiles?	NO	YES	SOMETIMES	NO	YES	SOMETIMES

Students 1 and 4, both male, were more reluctant in the use of the tiles initially compared to the girls in the group. They stopped using the tiles at a much earlier stage in the lessons. The four girls in the group were much more reliant on the tiles, with students 2 and 5, making full use of them during the test and explanations. Students 3 and 6 were beginning to start to do some of the questions without the tiles, but resorting back to them if they found the question difficult.

During the intervention

While the class were working with the tiles their results in both class work and homework were very high. I observed some excellent learning and engagement from all students and they were all keen to demonstrate the tiles to other teachers who came into the classroom.

A number of students were able to apply their knowledge to more difficult questions (beyond the curriculum) without any help or support from me, for example 3 students correctly solved the equation;

$$2x + 1 + 3x - 4 = 4x + 2 - 2x + 4$$

This demonstrates their ability to apply their knowledge to unfamiliar situations and questions. This shows flexibility, which is an indicator that they have some conceptual understanding.

Conclusion and Evaluation

My findings in lessons and speaking to the students, was that their conceptual understanding and ability to attempt some difficult algebraic questions had improved. The paired t-test result demonstrates that the class mean average had increased following the use of the tiles, but as the sample size is very small this needs to be treated with caution. The stimulated recall interviews demonstrated that some of the students had developed a better understanding, but this was not obvious for all of the students. Perhaps the time that had lapsed between the teaching using the tiles and the follow up test meant that a number of the students had forgotten how to use the tiles, or had lost some confidence with them. Given that this was a low achieving group who were not confident with Maths, this is likely to have been the case. In hindsight I would have given them another test immediately after they had learnt the tiles as well, so that these results and interviews could also be compared.

A large number of responses in the interviews were either "I don't know" or "I guessed". I think that this may have been because the students were worried about getting it wrong, or embarrassed to say anything if they were not totally confident in their own minds. Perhaps if someone they did not know so well, or wish to please, had conducted the interviews, they may have given a greater insight into their thought processes.

I thoroughly enjoyed conducting this action research and feel that I gained a greater understanding of how children learn. The tiles worked very well for this group and it is certainly an approach that I will continue with when introducing Algebra. I hindsight I would have planned the series of lessons more carefully and ensured that the way the students were using the tiles was the same way that they would be doing the written methods. A few of the students used the tiles in a slightly different manner, which I failed to pickup on. This then caused slight confusion when we moved to sketches and pencil and paper methods.

I remain concerned that some of the students have simply adopted a procedural approach to using the tiles. Although some were able to move towards the abstract concepts, there were still a few who I feel were not fully appreciating the link between the tiles and the abstract, and their conceptual understanding was still very limited. Given that the tiles can be used to introduce adding and subtracting negative numbers, I feel that in future I would introduce the tiles at an earlier stage in the students learning. This could then have the effect of encouraging the students to see the links between arithmetic and algebra.

The next step for this group would be to develop the use of the tiles for more difficult concepts and also help ensure that the students have scaffolding in place to help them to move toward pencil and paper methods.

If I were to conduct this research again I would also make at least some of the test not multiple choice. It was demonstrated when I spoke with the students that some had simply guessed the correct answer without any understanding and this will give some false positive marks.

Due to time constraints, and coverage of syllabus, I was also unable to complete a further cycle of action research with this group. This is a constant struggle, as a teacher we are faced with the battle of coverage of the topics needed to pass an exam versus the time needed to cover the topics in depth to ensure a relational understanding.

Trying to understand the children's learning has also helped me understand myself as both a teacher and a learner. One thing I did notice during this was how little the students questioned during the lessons, just accepting what they were told. Given that this is a key step in developing relational understanding, one of my next steps will be encouraging the students to question what they are learning.

Laurence Stenhouse (1975) said that good classrooms are those in which questions are asked to which no one knows the answer. I think the same can be said about good teachers, we must continue to ask ourselves the questions that do not have easy answers.

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